LexGuard — Full Foundations

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# BLOCK 1 — Parts I–V: GP, DRO, Lyapunov+SoCPM, KKT/Margins, Domain Exemplars

## PART I — GEOMETRIC PROGRAMMING (GP) DERIVATION FOR PARS-BUDGETED HARMONY

We maximize Harmony while budgeting (taxing or capping) risk via PARS and related factors. GP solves multiplicative structures globally after a log transform.

### P1.1 Setup (Posynomial Form)

H = (Benefit - Burden) - Safety\_Tax  
Safety\_Tax = alpha\*PARS + beta\*Gap + gamma\*Fragility  
PARS = kappa \* Hz \* Ex \* Vu \* (1 - Mt) with (Hz, Ex, Vu, 1-Mt) > 0  
Benefit = c0 \* Q / U with Q = I \* P (bilinear lift of UOE)  
  
Approach A (Budgeted):  
 maximize Q / U  
 subject to kappa\*Hz\*Ex\*Vu\*(1-Mt) <= R  
 Gap <= G  
 Fragility <= F  
 box bounds on (I, P, U, Hz, Ex, Vu, Mt)  
  
Approach B (Penalized):  
 maximize S = (Q/U) / [(kappa\*Hz\*Ex\*Vu\*(1-Mt))^alpha \* Gap^beta \* Fragility^gamma]  
(We proceed with A.)

### P1.2 GP-Eligibility

Standard GP: maximize monomial or minimize posynomial, with posynomial inequalities.  
Objective Q/U is monomial if Q = I\*P and variables are positive.  
PARS budget is a monomial <= constant (GP-compatible). Gap/Fragility upper bounds must be posynomial or bounded by posynomials.  
Standing positivity: I,P,U,Hz,Ex,Vu,(1-Mt) > 0.

### P1.3 Log-Convex Reformulation

Let y = log x for positive x.  
Monomial c\*Π x\_i^{a\_i} -> log c + Σ a\_i y\_i (affine)  
Posynomial Σ c\_k\*Π x\_i^{a\_ik} -> log Σ exp(log c\_k + Σ a\_ik y\_i) (log-sum-exp convex)  
Thus, maximize monomial <=> minimize negative affine in y (convex). Constraints become convex after log.

### P1.4 KKT System (Sketch)

Variables: z=log I, p=log P, u=log U, h=log Hz, e=log Ex, v=log Vu, s=log(1-Mt).  
L = -(z+p-u) + lambda\_R\*(log kappa + h+e+v+s - log R)  
 + lambda\_G\*(phi\_G(y) - log G) + lambda\_F\*(phi\_F(y) - log F)  
Stationarity equates marginal objective gains with weighted marginal risk increments; complementary slackness and feasibility complete KKT.  
KKT multipliers become risk shadow prices; align with Harmony weights in penalized form.

## PART II — DISTRIBUTIONALLY-ROBUST (DRO) DUALS: MOMENT & WASSERSTEIN

Goal: max\_x inf\_{P in Pset} E\_{theta~P}[ V(x; theta) ]

### D2.1 Moment-Based Ambiguity (Mean/Second Moment)

Theorem D2.1 (Linear-moment dual):  
 inf\_{P: E[phi]=m} E\_P[V] = sup\_{lambda} { lambda^T m + inf\_theta ( V(x;theta) - lambda^T phi(theta) ) }  
Corollary (Bands): moment bands add L1 penalties on lambda (support function of the band).

### D2.2 Wasserstein Ambiguity

Theorem D2.2 (Kantorovich/Lipschitz bound): If V is L\_c-Lipschitz under transport cost c,  
 inf\_{P: W\_c(P,Phat) <= eps} E\_P[V] = E\_{Phat}[V] - L\_c\*eps  
General dual: sup\_{lambda>=0} { E\_{Phat}[ inf\_theta ( V(theta) + lambda\*c(theta,tilde\_theta) ) ] - lambda\*eps }

## PART III — LYAPUNOV-STYLE CERTIFICATES COUPLING V\* DYNAMICS & SoCPM

Dynamics: V\*\_{t+1} = ((I\_t\*P\_t - (W\_t+eps))/U\_t)\*E + F\_t with |F\_t|<=F\_U, 0<=E<=1  
Safe set S = { x: H(x) >= 0, min\_i M\_{g\_i}(x) >= delta }  
SoCPM Redirect applies policy K(x\_t) when x\_t ∉ S.

### L3.2 Candidate Lyapunov and Negative Drift

L(x) = a\*[H(x)]\_- + b\*[delta - min\_i M\_{g\_i}(x)]\_+ + c\*PARS(x)  
Assume Redirect improves H, margins, and shrinks PARS by factor rho<1, and F\_t bounded.  
Theorem L3.1: ∃ a,b,c>0 s.t. L(x\_{t+1}) - L(x\_t) <= -eps\_L < 0 whenever x\_t ∉ S. => Safety invariance.

## PART IV — KKT/DUAL INTERPRETATION OF MARGIN MAPS AS SHADOW PRICES

Problem: maximize H(x) s.t. g\_i(x) >= 0.  
KKT: ∇H = Σ\_i lambda\_i\*∇g\_i, lambda\_i >= 0, lambda\_i\*g\_i=0.  
Margin M\_{g\_i}(x) = g\_i(x)/||∇g\_i(x)||.  
Theorem K4.1: marginal Harmony gain per unit increase in i-th margin equals lambda\_i (shadow price).

## PART V — DOMAIN EXEMPLARS (NUMERIC, END-TO-END)

V5.1 Legal/Compliance: I=4,P=0.7,W=1,U=2 => V=0.9; PARS=0.3; Tax=0.54; Burden=0.1 => H=0.26.  
SoCPM thresholding shows tone/mitigation trade for Proceed/Redirect.  
  
V5.2 Healthcare: V=0.6; initial H<0; increase Mt->0.8 => H>0; clears SoCPM.  
  
V5.3 Climate: V=0.333; initial H<0; Mt 0.3->0.8 => H>0; SoCPM Proceed.

# BLOCK 2 — Five Additional Math Sections + Consolidation

## A1) Multi-PARS Portfolio via GP with Couplings

PARS\_k = kappa\_k\*Hz\_k\*Ex\_k\*Vu\_k\*(1-Mt\_k)  
S\_k = (Q\_k/U\_k) \* Π\_j Penalty\_{k,j}^{-w\_{k,j}} , Q\_k=I\_k\*P\_k  
maximize Π\_k S\_k^{omega\_k}  
subject to Σ\_k PARS\_k <= R\_tot, Σ\_k Gap\_k <= G\_tot, Σ\_k Fragility\_k <= F\_tot,  
and couplings: Σ\_k c\_k\*(1-Mt\_k) <= B, Σ\_k I\_k <= I\_tot.  
After log: affine objective + log-sum-exp constraints ⇒ convex. KKT multipliers are portfolio shadow prices.

## A2) Wasserstein-2 DRO with Lipschitz Moduli

Setup: max\_x inf\_{P: W2(P,Phat)<=eps} E\_P[V(x;theta)], with cost ||theta-theta'||\_2^2.  
Theorem A2.1: If V is L-Lipschitz in theta, inf\_{W2<=eps} E[V] >= E\_{Phat}[V] - L\*eps.  
Dual form: sup\_{lambda>=0} { E\_{Phat}[ inf\_theta ( V + lambda\*||theta-tilde||^2 ) ] - lambda\*eps^2 }.

## A3) Composite Lyapunov with Tone (Ar) & Exposure (Ex) for De-escalation

L(x) = a\*[H]\_- + b\*[delta - min\_i M\_g]\_+ + c\*PARS + d\*psi(Ar) + e\*chi(Ex)  
Redirect ensures: H↑ by eta\_H, margins↑ by eta\_M, PARS\*(t+1) <= rho\*PARS\_t, Ar\_{t+1} <= rho\_Ar\*Ar\_t, Ex\_{t+1} <= rho\_Ex\*Ex\_t.  
Theorem A3.1: ∃ weights => ΔL <= -eps\_c < 0 outside S ⇒ formal de-escalation.

## A4) Second-Order Sensitivity (Parametric LP Bands / Smooth Hessians)

Within a stable LP basis B\*: J(θ) = J(θ0) + Σ\_i π\*\_i Δθ\_i.  
Band: J ∈ [J0 - Σ|π\*\_i||Δθ\_i|, J0 + Σ|π\*\_i||Δθ\_i|].  
Basis flips capture curvature; smooth surrogate (softplus on McCormick) yields Hessians if needed.

## A5) Multi-Period Worked Case with Uncertainty Sets (Auditor-Ready)

For t=1..T: V\_t=(I\_t\*P\_t - W\_t)/U\_t; PARS\_t=kappa\*Hz\_t\*Ex\_t\*Vu\_t\*(1-Mt\_t);  
H\_t = (V\_t - Burden\_t) - [alpha\*PARS\_t + beta\*Gap\_t + gamma\*Fragility\_t].  
Objective: J = inf\_{θ\_t in U\_t} Σ δ^{t-1} H\_t(x\_t; θ\_t), with SoCPM policy.  
Certificate pack: LP/GP logs, DRO certs, Lyapunov drift, margin series, Redirect traces.

## B) Consolidation Snapshot

Covered (12 sections): UOE; Harmony; PARS; SoCPM; ILC; Gap/Tethered; Margin Maps; Portfolio GP; DRO; Composite Lyapunov; 2nd-Order Sensitivity; Multi-Period Case.  
Missing (now addressed in Block 3): Provenance/Lineage; SBOM Gate; Traceability; Adversarial calculus; Calibration; Complexity; Implementation correctness; Legal alignment.

# BLOCK 3 — Sections 13–20 Fully Written (Governance & Ops)

## 13) Provenance & Lineage Ledger — Formal Model and Guarantees

Objects and commits:  
 Capsule C = (DataVer, ModelVer, GuardrailVer, Seed, Config, Runtime)  
 Commit(a) = H( bytes(a) || C || Parents(a) )  
Ledger: DAG of commits with Merkle forest for O(log N) inclusion proofs.

Theorem 13.1 (Minimal blame landing): Build auxiliary s-t flow on lineage DAG; a minimum cut intersects all root→incident paths ⇒ minimal landing set; polynomial time.

Theorem 13.2 (Binding): Collision-resistant hash => cannot alter a or C without changing Commit(a).

## 14) SBOM Gate for AI — Release Policy as Mathematics

SBOM S(r) = (ModelVer, DataSlices, GuardrailSet, EvalSuite, RiskBudget).  
Gate rules: signatures; Σ PARS(a) <= R\_tot; eval metrics >= thresholds τ\_m; calibrated T for SoCPM.

Theorem 14.1 (Ex-ante/ex-post soundness): If gate holds and SoCPM uses calibrated T, shipped artifacts either satisfy H>=0 on validation or are redirected at runtime.

Theorem 14.2 (Calibration ROC band): ∃ [T\_-,T\_+] yielding monotone trade between false-blocks and false-passes.

## 15) Requirements Traceability — Spec → Proofs → Tests → Metrics

Trace matrix R maps each requirement to: theorems (T), tests (X), metrics (M) with ledger pointers.

Theorem 15.1 (Audit completeness): If every requirement has linked theorem + passing tests + metrics within tolerances, the system is audit-complete.

## 16) Adversarial Model & Red-Team Calculus

Threat sets Δ\_prompt, Δ\_eval, Δ\_data, Δ\_jail (compact). Robust objective uses tethered value:  
 Ṽ(x) = inf\_{δ∈Δ} V(x;δ) ; H̃(x) = (Ṽ - Burden) - Safety\_Tax.

Theorem 16.1 (Existence): Under compactness/continuity on compact feasible set, max\_x Ṽ and max\_x H̃ exist.

Theorem 16.2 (Coverage lower bound): With coverage ρ and Lipschitz V, inf\_Δ V ≥ inf\_S V − L\*(1−ρ)\*diam(Δ).

## 17) Calibration Protocol — (alpha, beta, gamma, T) from Data

Fit weights via NNLS: E[L] ≈ alpha\*E[PARS] + beta\*E[Gap] + gamma\*E[Fragility].  
Theorem 17.1: NNLS preserves monotonicity ⇒ Harmony nonincreasing in risk components.  
Threshold T chosen by validation ROC; PAC-like bound with VC-dim d and n samples:  
 TrueError(T) ≤ EmpError(T) + O( sqrt( (d\*log n + log(1/δ)) / n ) ).

## 18) Complexity & Runtime — Scaling Laws

Lifted UOE LP: polynomial time (interior-point or simplex).  
Portfolio GP: convex, ~O(p^3) with sparsity benefits.  
DRO duals: small multiplier dimension overhead.  
Multi-period: ~linear in T when separable; coupling adds convex constraints.

## 19) Implementation Correctness — Capsules, Numerics, Tolerances

Capsule: pinned container, code/weights/data digests, seeds, solver versions/tolerances.  
Theorem 19.1: Reproducibility up to epsilon for Lipschitz functionals with fixed tolerances.  
Guard against silent violations: interval arithmetic on guardrails; accept only if lower bound >= 0.  
Theorem 19.2: Interval evaluation prevents false passes due to rounding.

## 20) Legal Alignment Appendix — ALARP & Duty of Care

ALARP mapping: maximize H = (Benefit - Burden) - alpha\*PARS - beta\*Gap - gamma\*Fragility implies mitigation increases until priced marginal risk reduction equals marginal burden.

Duties: Runtime (SoCPM), Ex-ante (SBOM Gate), Ex-post (Lineage/Blame landing). Together => measurable, auditable standard of care.